Correspondences of probability measures with restricted marginals revisited[1]

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Summary. We reprove a result of J. Bergin on the continuity of the correspondences of probability measures on a product space with restricted marginals. The proof is based on some functorial properties of probability measures. It works also for arbitrary number of coordinate spaces and for non-metrizable case.

Keywords and Phrases. Probability measures on product spaces, Continuity of correspondence, Bicommutative diagram.

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1. Introduction

In [1] it is proved that the correspondence assigning to every probability measures on two coordinate spaces the set of probability measures with these marginals is continuous. The proof of this result in [1] is rather technical. Here we develop a different approach to this question and use some known properties of the probability measure functor as building blocks of our proof. This proof works smoothly only in compact case, however, our result is not contained in that of [1], because we do not restrict the number of coordinate spaces. Different problems of economics and game theory are related to the properties of the mentioned correspondence of probability measures with restricted marginals for the case of arbitrary finite number of coordinate spaces. Here we only mention one such problem, briefly discussed in the introduction of [1]. Consider income distributions at the time period k as probability measures μ_k on a space Y of possible incomes. Then any redistribution policy can be interpreted as a probability measure, τ, on the product $Y^k = Y \times \cdots \times Y$ such that the marginal distributions of τ are μ_i, $i = 1, \ldots, k$ and the problem arises of welfare maximization for prescribed sequence $\mu_1, \ldots, \mu_k$ and dependence of this maximum on $\mu_1, \ldots, \mu_k$.

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Some methods involved in the proof are closely connected to investigations of compact spaces by representing them as the limits of inverse systems, systematically performed by E. Shchepin [2]. Shchepin's result on bicommutativity of open functors is one of the main ingredients of the proof. One of the goals of this note is to make this powerful technique familiar for economists.

2. Preliminaries

Given a product $\prod_i X_i$, we always denote by π_i the projection onto the ith coordinate. By 1_X the identity map of X is denoted.

2.1. **Correspondences and open maps.**

Given a topological space X denote by 2^X its *hyperspace*, i. e. the set of all nonempty compact subsets endowed with the *Vietoris topology*. A base of this topology consists of the sets of the form

$$\langle V_1,\dots,V_n\rangle = \{A\in 2^X \mid A\subset \cup_{i=1}^n V_i,\ A\cup V_i\neq\emptyset \text{ for all } i\},$$

where $V_1,\dots,V_n$ run through the topology of X. Given a compact-valued correspondence $F\colon X\to Y$, we can regard it as a (single-valued) map, f, from X to 2^Y and continuity of the correspondence F is equivalent to continuity of f whenever 2^Y is endowed with the Vietoris topology.

Every continuous onto map $f\colon X\to Y$ determines the *inverse* map $f^{-1}\colon Y\to 2^X$, $y\mapsto f^{-1}(y)$. It is a well-known fact that an onto map f is open if and only if the map f^{-1} is continuous.

2.2. **Bicommutative diagrams.**

A commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "u"] \\ Z \arrow[r, "v"'] & T \end{tikzcd} \tag{2.1}$$

is called *bicommutative* (see, e. g. [5]) if its *characteristic map* $\chi = (f,g)\colon X\to Y\times_T Z = \{(y,z)\in Y\times Z \mid u(y)=v(z)\}$ is onto. The following lemma is proved by Shchepin [2].

Lemma 2.1. *Suppose that in diagram (2.1) X,Y,Z,T are compact, f,g,u,v are continuous and g,u are onto. If f is an open map, then so is v.*

2.3. **Inverse systems.** An *inverse sequence* $\mathcal{S} = (A_i, f_i)$ consists of topological spaces A_i and bonding maps $f_i\colon A_{i+1} \to A_i$. The *inverse limit* $\lim \mathcal{S}$ consists of the sequences (a_i) in the product $\prod A_i$ satisfying the condition: $a_i = f_i(a_{i+1})$, for every i. A *morphism* of an inverse sequence $\mathcal{S} = (A_i, f_i)$ into an inverse sequence $\mathcal{S}' = (A_i', f_i')$ is a sequence of maps $(g_i\colon A_i \to A_i'$ such that $f_i' g_{i+1} = g_i f_i$, for all i. Obviously, such a morphism (g_i) uniquely generates the map $\lim(g_i)\colon \lim \mathcal{S} \to \lim \mathcal{S}'$ by the formula $\lim(g_i)(a_i) = (g_i(a_i))$.

Lemma 2.2. *Suppose that all the maps of a morphism $(g_i)\colon \mathcal{S} \to \mathcal{S}'$ are open and all the square diagrams*

$$\begin{tikzcd}
A_{i+1} \arrow[r, "g_{i+1}"] \arrow[d, "f_i"'] & A'_{i+1} \arrow[d, "f'_i"] \\
A_i \arrow[r, "g_i"'] & A'_i
\end{tikzcd}$$

are bicommutative. Then the map $\lim(g_i)$ is open.

For the proof see [2]. In the sequel, such a morphism for which the diagrams from Lemma 2.2 are bicommutative is called *bicommutative.*

2.4. **Probability measures and bicommutative diagrams.** By P we denote the probability measure functor. The latter means that P acts not only on spaces but also on maps preserving the composition and the identity map. We will use the fact that P is a *bicommutative* functor in the sense that it preserves the class of bicommutative diagrams (see [2]).[2]

Lemma 2.3. *For arbitrary maps $f_i\colon X_i \to X_i'$, $i = 1, \dots, k$, the diagram*

$$\begin{tikzcd}
P(\prod X_i) \arrow[r, "M_{X_1,\dots,X_k}"] \arrow[d, "P(\prod f_i)"'] & \prod PX_i \arrow[d, "\prod Pf_i"] \\
P(\prod X_i') \arrow[r, "M_{X_1',\dots,X_k'}"'] & \prod PX_j'
\end{tikzcd} \tag{2.2}$$

is bicommutative.

[2]In [2] the bicommutativity of the functor P is derived from the fact that P is *open*, i. e. preserves the class of open maps (Ditor–Eifler [4]). As remarked in [1], in the compact case, the continuity of the correspondence $\mu \mapsto \{\nu \in P(X \times Y) \mid P\pi_1(\nu) = \mu\}$ from PX to $P(X \times Y)$. Here is an alternative proof: the projection $\pi_1\colon X \times Y \to X$ is an open map and, by the cited theorem of Ditor and Eifler, the map $P\pi_1\colon P(X \times Y) \to PX$ is also open, which is equivalent to the continuity of the above correspondence.

Proof. Given $\tau' \in P(\prod X_i')$ and $(\mu_1, \dots, \mu_k) \in \prod PX_i'$ such that

$$M_{X_1',\dots,X_k'}(\tau') = \prod Pf_i(\mu_1, \dots, \mu_k) = (Pf_1(\mu_1), \dots, Pf_k(\mu_k))$$

we proceed as follows.

For every $j \le k$ denote by $\mathcal{D}_j$ the diagram

$$\begin{tikzcd} \prod_{i\le j} X_i \times \prod_{i>j} X_i' \arrow[r, "\pi_j"] \arrow[d, "\prod_{i\le j} f_i \times 1_{\prod_{i>j} X_i'}"'] & X_j \arrow[d, "f_j"] \\ \prod X_i' \arrow[r, "\pi_j"'] & X_j' \end{tikzcd}$$

which is obviously bicommutative.

Since $P\pi_1(\tau') = Pf_1(\mu_1)$, applying the functor P to the diagram $\mathcal{D}_1$ we find $\tau_1 \in P\left(X_1 \times \prod_{i>1} X_i'\right)$ such that

$$P\pi_1(\tau_1) = \mu_1,\ P\left(f_1 \times 1_{\prod_{i>1} X_i'}\right)(\tau_1) = \tau'.$$

Consider natural l, $1 \le l \le k$, and suppose that for every $j < l$ we have defined $\tau_j \in P\left(\prod_{i\le j} X_i \times \prod_{i>j} X_i'\right)$ such that $P\pi_j(\tau_j) = \mu_j$ and $P\left(\prod_{i\le j} f_j \times 1_{\prod_{i>j} X_i}\right)(\tau_j) = \tau_{j-1}$. Note that

$$\begin{aligned} Pf_l(\mu_l) =& P\pi_l(\tau') = P\pi_l\left(P\left(f_1 \times 1_{\prod_{i>1} X_i'}\right)\right) \\ =& \dots \\ =& P\pi_l\left(P\left(f_1 \times 1_{\prod_{i>1} X_i'}\right)\dots P\left(\prod_{i\le l-1} f_i \times 1_{\prod_{i>l-1} X_i'}\right)\right)(\tau_{l-1}) \\ =& P\pi_l(\tau_{l-1}). \end{aligned}$$

Applying the functor P to the bicommutative diagram $\mathcal{D}_j$ we conclude that there exists $\tau_l \in P\left(\prod_{i\le l} X_i \times \prod_{i>l} X_i'\right)$ such that $P\pi_l(\tau_l) = \mu_l$ and $P\left(\prod_{i\le l} f_i \times 1_{\prod_{i>l} X_i'}\right)(\tau_l) = \tau_{l-1}$.

It is easy to see that $\tau = \tau_k$ has the following properties: $M_{X_1\dots X_k}(\tau) = (\mu_1, \dots, \mu_k)$ and $P(\prod f_i) = \tau'$. This proves the bicommutativity of diagram (2.2). $\square$

3. Result

Theorem 3.1. *Let $X_1, \dots, X_k$ be a finite sequence of compact spaces. Then the multivalued map assigning to every $\mu_1, \dots, \mu_k$, where $\mu_i \in PX_i$, for every i, the set*

$$\begin{aligned} M(\mu_1, \dots, \mu_k) =& M_{X_1,\dots,X_k}(\mu_1, \dots, \mu_k) = \{\nu \in P\left(\prod X_i\right) \\ & \mid P\pi_i(\nu) = \mu_i,\ i = 1, \dots, k\} \end{aligned}$$

is continuous.

Proof. It consists of the following steps. We first prove the statement for finite $X_1, \dots, X_k$. Next, we consider zero-dimensional spaces $X_1, \dots, X_k$.

1) Suppose that $X_1, \dots, X_k$ are finite. Then the map $M_{X_1,\dots,X_k}$ is an affine surjective map of compact convex polyhedra. In order to prove that the latter, say $f\colon A \to B$ is open, it suffices to show that arbitrary point a of A is in the image of a selection of f. Denote by C the union of simplices of the boundary of B that do not contain the point $f(a)$. For every vertex c of C let $g(c)$ be an arbitrary point of $f^{-1}(c)$. Extend the so defined map g onto C affinely on every simplex of C. Finally, every point b in B can be uniquely represented in the form $tf(a) + (1-t)c$, where $c \in C$. Define $g(b) = t(a) + (1-t)g(c)$. We see that $fg = 1_B$ and $a \in g(B)$.

2) Suppose now that the spaces $X_1, \dots, X_k$ are zero-dimensional (recall that a compact metric space is *zero-dimensional* if it posesses a base of sets that are simultaneously open and closed). It is well-known that then there exist inverse sequences $\mathcal{S}_j = \{X_{ij}, f_{ij}\}$ such that all spaces X_{ij} are finite and $\lim \mathcal{S}_j = X_j$, $j = 1, \dots, k$. Then $P(\prod X_j) = \lim\{P(\prod_j X_{ji}), P(\prod_j f_{ji})\}$ and

$$\prod PX_j = \lim\{\prod_j PX_{ji}, \prod_j Pf_{ji}\}.$$

The sequence $(M_{X_{1j},\dots,X_{kj}})_j$ forms a bicommutative morphism of the inverse sequence $\{P(\prod_j X_{ji}), P(\prod_j f_{ji})\}$ to the inverse sequence $\{\prod_j PX_{ji}, \prod_j Pf_{ji}\}$. Using the fact that the functor P is continuous, i. e. commutes with the operation of limit of inverse sequence [3], it is easy to see that $M_{X_1,\dots,X_k} = \lim(M_{X_{1j},\dots,X_{kj}})_j$. As we already proved, the maps $M_{X_{1j},\dots,X_{kj}}$ are open. By Lemma 2.2, the map $M_{X_1,\dots,X_k}$ is also open.

3) Finally, suppose that $X_1, \dots, X_k$ are arbitrary compact metrizable spaces. There exist zero-dimensional compact metrizable spaces $X'_1, \dots, X'_k$ and continuous onto maps $f_i\colon X_i \to X'_i$, $i = 1, \dots, k$. As we have already proved, the map $M_{X'_1,\dots,X'_k}$ is open. Applying consequently Lemmas 2.3 and 2.1 we conclude that $M_{X_1,\dots,X_k}$ is open as well. $\square$

4. Remarks

One can generalize the main result in two directions. As it was told in the introduction, in Theorem 3.1, $X_1, \dots, X_k$ need not be metrizable. To prove the theorem in this case, we follow the line of the proof of Theorem 3.1, however, one should consider more general (not necessarily countable) inverse systems instead of inverse sequences; see [2] for details.

Another generalization can be obtained if we consider arbitrary family of coordinate spaces, not necessarily finite. The given proof requires only some minor changings.

Note also that for the case of measures with finite supports the continuity of correspondences with restricted marginals was also proved in [6]. As for the measures with supports of cardinality that is bounded from above, we have the following. Let P_nX denotes the space of measures in PX whose supports are of cardinality $\leq n$ (i. e. any measure, μ, in P_nX is of the form $\mu = \sum_{i=1}^n \alpha_i\delta_{x_i}$, where $\alpha_i \geq 0$, $\sum_{i=1}^n \alpha_i = 1$, δ_{x_i} is the Dirac measure concentrated in $x_i \in X$. Then the multivalued map assigning to every $\mu \in P_nX$, $\nu \in P_nY$ the set of measures in $P_n(X \times Y)$ with marginal distributions μ, ν, in general, is not continuous for all $n \geq 2$.

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